$\star \forall x \in \left[0;\frac{\pi}{4}\right]; \varphi'(x) = [\psi(tgx)]' - [\psi(0)]'$

$= (tgx)'.\psi'(tgx) = (1+tg^2x)\frac{tg^2x+2}{tg^2x+1} = tg^2x + 2.$

b- $\forall t \in \left[0;\frac{\pi}{4}\right]; \varphi'(t) = tg^2t + 2.$ et $\varphi(0) = \int_0^{tg0} f(t)dt = 0$

donc φ est la primitive de $(t \mapsto tg^2t+2)$ sur $\left[0;\frac{\pi}{4}\right]$ qui s'annule en 0

$\Rightarrow \forall x \in \left[0;\frac{\pi}{4}\right]; \varphi(x) = \int_0^x (tg^2t+2)dt = \int_0^x (1+tg^2t+1)dt$

$= [tgt + t]_0^x = tgx + x.$

c- $A = \left[\int_0^1 (f(x)-1)dx\right]$ u.a. (u.a.: unité d'aire)

$= \left[\int_0^1 f(x)dx - \int_0^1 dx\right]$ u.a.

$= \left[\int_0^{tg\frac{\pi}{4}} f(x)dx - 1\right]$ u.a.

$= \left[tg\frac{\pi}{4} + \frac{\pi}{4} - 1\right]$ u.a. $= \frac{\pi}{4}$ u.a.

EXERCICE 2

1/a- $\forall x \in \left[0;\frac{\pi}{4}\right]$ et $\forall n \in IN, \left[(tgx)^{n+1}\right]' = (n+1)(1+tg^2x)(tgx)^n$

b- $\forall n \in IN, I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} (tgx)^n dx + \int_0^{\frac{\pi}{4}} (tgx)^{n+2} dx$

$= \int_0^{\frac{\pi}{4}} (tgx)^n (1+tg^2x)dx$

$= \frac{1}{n+1}\int_0^{\frac{\pi}{4}} (n+1)(tgx)^n(1+tg^2x)dx$

$= \frac{1}{n+1}\int_0^{\frac{\pi}{4}} \left[(tgx)^{n+1}\right]' dx$

$= \frac{1}{n+1}\left[(tgx)^{n+1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n+1}.$

2/a- $\forall n \in IN, \forall x \in \left[0;\frac{\pi}{4}\right]$ on a $(tgx)^n \geq 0.$

$\Rightarrow \forall n \in IN,$ on a $I_n = \int_0^{\frac{\pi}{4}} (tgx)^n dx \geq 0.$

• $\forall n \in IN,$ on a $0 \leq I_{n+2}$

$\Leftrightarrow \forall n \in IN,$ on a $0 + I_n \leq I_{n+2} + I_n$

$\Leftrightarrow \forall n \in IN,$ on a $I_n \leq \frac{1}{n+1}.$

b- $\forall n \in IN, 0 \leq I_n \leq \frac{1}{n+1}$ de plus $\lim_{n\to+\infty} \frac{1}{n+1} = 0$

$\Rightarrow \lim_{n\to+\infty} I_n = 0.$

3/a- $\forall x \in \left[0;\frac{\pi}{4}\right]; g'(x) = [Log(\cos x)]' = \frac{(\cos x)'}{\cos x} = \frac{-\sin x}{\cos x} = -tgx$

b- $* \; I_1 = \int_0^{\frac{\pi}{4}} tgxdx = -\int_0^{\frac{\pi}{4}} -tgxdx = -[Log(\cos x)]_0^{\frac{\pi}{4}}$